A Quantum Algorithm for the Prediction Step of a Bayesian Recursion

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***Abstract*—The prediction step is a crucial element of the Bayesian recursion for target tracking and state estimation in general. Discrete representations of the probability density function (pdf) can deal with non-linear models and non-Gaussian noise, however, the prediction step is challenging to solve on classical computers. In this paper, a novel concept of quantum simulation to solve the application of a continuous noise motion model to a pdf is presented. The pdf is prepared as the squared amplitudes for the basis states spanned by the number of used qubits. The number of required qubits grows linearly in the number of time steps to simulate in a prediction phase. One step performs a single Brownian motion, which is used to generate the diffusion of the Wiener increments. A drift function can be implemented based on a separate register, which holds the pdf on the velocity information for an adequate discretization. The approach will be visualized in terms of quantum circuits and evaluated based on a quantum simulator.**

I. Introduction

Traditional computational approaches, while effective to a certain extent, often grapple with the intrinsic complexities and high-dimensional nature of real-world data, particularly when dealing with random processes. The inherent limitations of classical computing in processing such stochastic phenomena necessitate a paradigm shift towards more capable and efficient computational frameworks. In this context, quantum computing emerges as a technology with the potential to revolutionize data processing, offering significant advantages over its classical counterparts through phenomena such as superposition and entanglement.

This paper introduces a novel quantum computing algorithm designed for the transformation of a discrete *probability density function* (pdf) according to continuos noise state transition models for object tracking and state estimation. In a Bayesian recursion of *prediction* and *filtering* for an iterative processing of noisy observations from one or multiple sensors, this corresponds to the prediction step, where time passes and the pdf reflects the fact, that unknown distortion is happening to the random process. The algorithm leverages the unique capabilities of quantum mechanics to simulate the drift and diffusion of random variables using the superposition property of multi-qubit systems and thereby enabling a efficient and exact solution even in high dimensions. By employing quantum superposition, the presented approach allows a simultaneous representation of multiple states of a random process. The amplitudes of the corresponding wave function can directly be transformed in to probabilities by taking their square of their absolute values. They can also be observed by repeated measurements of the quantum circuit.

The conventional methods for the solution of object dynamics in discrete state spaces often suffer from computational inefficiency and scalability issues. In the recent past, tracking algorithms using tensor decompositions have been proposed to overcome this problem [1]. Based on a discretization of the state space, the pdf becomes a multi-way tensor. The solution of the prediction step is obtained by solving the *Fokker–Planck Equation* (FPE), which is equivalent to the integral formulation known as the *Chapman–Kolmogorov Equation*. Challa and Bar-Shalom for instance use finite differences for small-dimensional examples in [2] to obtain the solution of the FPE and show that a consistent result is obtained even for highly non–linear problems with large noise variances. Solving the FPE for high-dimensional problems is computationally challenging. Current state-of-the-art solutions with tensor decompositions use a Taylor expansion of the tensor exponential, which suffers from a limited accuracy due to the bad convergence property.

Our quantum algorithm addresses these challenges by efficiently simulating the diffusion process of the random variable, thus providing a powerful tool for data fusion using discrete pdfs that significantly enhances the accuracy and reliability of the outcomes in non-linear and non-Gaussian scenarios.

The core of our proposed solution is a quantum algorithm that simulates the drift and diffusion process of random variables by exploiting the principles of quantum superposition. This approach not only accelerates the computational process but also is fully robust since the solution of the FPE is not required. We demonstrate the algorithm's efficacy through a series of experiments that highlight its potential to outperform traditional state transition methods for data fusion and state estimation, especially in scenarios characterized by high-dimensional data.

Structure: This paper is structured as follows. In Section II, some background information on the problem of the prediction step for discrete density functions is provided. Then, an introduction to quantum computation is given in Section III for readers who are not familiar with the matter. In Section IV our approach of quantum simulation of Brownian motion is presented in order to reflect a stochastic diffusion of a Wiener process as the resulting pdf. The next part in Section V is dedicated to the derivation of a drift operation, where the the density function for the velocity of the stochastic process is given in a separate quantum register. A numerical evaluation based on a quantum computer simulator is provided in Section VI. The paper closes with a conclusion in Section VII.

II. Formulation of the Problem

The theory of target tracking has exposed a growing family of algorithms to compute the probability density function (pdf) of a system state based on noise corrupted sensor observations. An estimate of the state is then obtained by taking the mean of the pdf[1]. The corresponding covariance matrix additionally provides a measure of accuracy for this estimate. Bayesian estimation is the framework of recursive filtering methodologies, which allow us to process a current measurement by means of a *prior* or *initial* density and a measurement likelihood function which statistically describes the performance of the sensor. Thus, a tracking algorithm is an iterative updating scheme for calculating a conditional pdf $p(\mathbf{x}_k|\mathcal{Z}^k)$ that represents all available knowledge on the object state $\mathbf{x}_k$ at some time t_k, which typically is chosen as the present time. The densities are explicitly conditioned on the sensor data time series $\mathcal{Z}^k$. The iterative scheme consists of two processing steps per update cycle, referred to as *prediction* and *filtering*.

Based on a suitable discretization of the state space to points $\left[\mathbf{x}_d^{(n_d)}\right]_{n_d}$ with a discretization size Δ_d for $d = 1,\dots,D$ and $n_d = 1,\dots,N_D$ the density function on the state variable $\mathbf{x}$ becomes a D-way tensor of dimension $N_1 \times \cdots \times N_D$:

$$p(\mathbf{x}) = [p([\mathbf{x}_d]_{n_d})]_{n_1,\dots,n_D} \in \mathbb{R}^{N_1\times\cdots\times N_D}. \tag{1}$$

Since this representation would require to compute and store $\prod_{d=1}^{D} N_d$ entries of the tensor, a number which is exponential in D, efficient data compression models have been applied for high dimensional problems [3], [1]. For instance, let us assume a target tracking problem with 2 dimensions in position (x and y) and velocity ($\dot{x}$ and $\dot{y}$), respectively. With a discretization of 100 points for all dimensions $d = 1,\dots,4$, the tensor has 10^8 entries. If fully expanded, this corresponds to ≈ 0.75 GB of RAM space. Operation matrices have the square size, which becomes infeasible for nowadays computer systems, since this would correspond to $\approx 5 \cdot 10^8$ GB data.

So, how can quantum computing help us? It is well-known that quantum algorithms profit from the superposition principle of qubits. A qubit is the fundamental unit of quantum information in quantum computing, analogous to a bit in classical computing. However, unlike a classical bit, which can be in one of two states (0 or 1), a qubit can exist in a superposition of these states. While one qubit covers two ground states in superposition, n qubits cover 2^n ground states. This exponential growth is one of the most important superpowers of quantum computation concepts. Considering the problem from above with 10^8 possible ground states, one can see that already 27 qubits are sufficient to represent the full state space. A reasonable splitting into 4 registers with 7 qubits each per dimension would work with 28 qubits only.

The state of a quantum circuit is defined by its wavefunction, described in greater detail below. The density function corresponds to its absolute value squared. In other words, one can use the superposition principle in quantum computing to represent a discrete probability density function, where the probability values can be obtained by repeated measurements. Since the wavefunction collapses after each observation, a high number of preparations of the qubits is required. However, statistics such as mean and covariance matrix may well be obtained by much less measurements with sufficient precision.

In this paper, we address the problem of the prediction step in a Bayesian recursion for a discrete pdf, which is represented in a prepared quantum circuit. Since the integral solution given by the Chapman–Kolmogorov Equation is ill-posed for high-dimensional discrete tensors, the approach of quantum simulation is proposed. To this end, it is assumed that the time evolution of the system is described by a continuous–time stochastic process given by

$$\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\mathrm{d}t + \mathbf{G}(\mathbf{x}, t)\mathrm{d}\mathbf{w}, \tag{2}$$

where $\mathbf{f}$ is the drift function, $\mathbf{G}$ is the matrix of all diffusion coefficients, and $\mathrm{d}\mathbf{w}$ are the increments of a multivariate Brownian motion with covariance $\mathrm{d}t\mathbf{I}$. Since the multi-dimensional case is easily obtained by multiple independent realizations we focus on the 1-D case with $\mathbf{G} = 1$.

Since the pdf representation of the previous posterior for time t_0 is given as a quantum state $|\psi_0\rangle$, our goal is to provide the means to use a given model in (2) in a quantum simulation in order to obtain the prior in a state $|\psi_1\rangle$ for time $t_1 = t_0 + \Delta_t$. If the drift and diffusion operators are given by corresponding gates, the core of the problem is to simulate Brownian motion in a register, which is then transformed to a diffusion of the modeled dimensions and strength.

[1] Depending on the scenario, for instance the expectation value, the maximum value, the median, or other statistics of the pdf can be used.

III. Introduction to Quantum Computation

For the wavefunction, that is the quantum state, two ground states of a qubit are defined by the states, which always return 0 and 1, respectively, when measured. These define the basis states denoted with $|0\rangle$ and $|1\rangle$. A qubit is typically represented as a linear combination (superposition) of its basis states, $|0\rangle$ and $|1\rangle$, which are often referred to as the computational basis. The general state $|\psi\rangle$ of a qubit can be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{3}$$

Here, α and β are complex numbers that represent the probability amplitudes of the qubit being in the $|0\rangle$ and $|1\rangle$ states, respectively. These amplitudes hold the key to the probabilistic nature of quantum mechanics. The probabilities of measuring the qubit in either state are given by the squares of the magnitudes of these amplitudes:

- Probability of measuring the state $|0\rangle$:

$$P(0) = |\alpha|^2. \tag{4}$$

- Probability of measuring the state $|1\rangle$:

$$P(1) = |\beta|^2, \tag{5}$$

where $|\alpha|^2 = \alpha^*\alpha = a^2 + b^2$ for $\alpha = a + ib$ and $a, b \in \mathbb{R}$ and we used the notation α^* for the complex conjugate $\alpha^* = a - ib$.

The total probability must always sum up to 1, enforcing the normalization condition for a single qubit given by

$$|\alpha|^2 + |\beta|^2 = 1. \tag{6}$$

Since the coefficients α and β are complex, they carry more information than the probability likelihood: Using the polar representation $\alpha = |\alpha|e^{i\varphi}$, we denote $\varphi \in [0, 2\pi]$ the *phase*. As indicated above, the phase implies interference effects, which can be used by quantum algorithms for efficient solutions.

It is possible to identify $|0\rangle$ and $|1\rangle$ as the basis vectors in a complex 2D vector space:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8}$$

The vector representation has the advantage, that all computations of qubit states reduce to vector algebra. In particular, one may write a given state $|\psi\rangle$ as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{9}$$

$$= \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{10}$$

$$= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{11}$$

Using the bra-ket notation, where a "bra" $\langle\psi| = |\psi\rangle^\dagger$ is a conjugate transpose of a "ket" vector, the probabilities can also be written as

$$P(0) = |\langle 0|\psi\rangle|^2 = |\alpha|^2, \tag{12}$$

$$P(1) = |\langle 1|\psi\rangle|^2 = |\beta|^2. \tag{13}$$

It must be noted that measuring the state fixes its outcome, that is, measuring it once or multiple times again will yield the exact same outcome. As a consequence, the state of the measured qubit has changed: Either $|\psi\rangle = |0\rangle$ or $|\psi\rangle = |1\rangle$ depending whether 0 or 1 has been observed. In quantum physics, this effect is denoted with a "collapsed wavefunction". Since the wavefunction encodes the probabilities in the state space, it is collapsed in the sense, that all probability mass has moved to a single state.

Assume, the state $|\psi\rangle$ is being measured and one obtains 0. As described above, the state then has changed to $|\psi\rangle = |0\rangle$. Therefore, in order to observe the distribution of single qubit or a register of multiple qubits, one has to run the same algorithm multiple times with a well-defined initial state which then yields *deterministically* the same end state. By means of multiple runs with a following observation one obtains the histogram of the outcomes which tends to the true distribution for the number of observations grow to infinity[2].

A multi qubit system written as $|\psi\rangle = |q_n q_{n-1} \cdots q_1\rangle$ with n qubits q_i for $i = 1, \ldots, n$ is described by the Kronecker product

$$|\psi\rangle = |q_n\rangle \otimes |q_{n-1}\rangle \otimes \cdots \otimes |q_1\rangle. \tag{14}$$

The $N = 2^n$ amplitude parameters of $|\psi\rangle$ are normalized such that the absolute values squared sum up to one. In particular, the N ground states $|\mathbf{j}\rangle$ are given by $|j_n j_{n-1} \cdots j_1\rangle$, where the j_i are the digits of the binary representation of $\mathbf{j} \in 0, \ldots, N-1$.

a) Example: Consider the case where one has $n = 2$ qubits, that is 4 ground states. These are given by

- $|\mathbf{0}\rangle = |00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
- $|\mathbf{1}\rangle = |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
- $|\mathbf{2}\rangle = |10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
- $|\mathbf{3}\rangle = |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

[2]Correlated noise is neglected here for the sake of simplicity.

A quantum state in this circuit is fully described by the complex coordinates $(a_{00}, a_{01}, a_{10}, a_{11}) \in \mathbb{C}^4$ and it holds that

$$|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1. \tag{15}$$

IV. Quantum Simulation of Brownian Motion and Diffusion

Brownian motion, a cornerstone concept in the realm of stochastic processes, epitomizes the erratic and unpredictable movement of particles suspended in a fluid medium. First observed by botanist Robert Brown in 1827, this phenomenon has transcended its initial biological context to become a fundamental model in mathematical physics, finance, state estimation of dynamic systems, and various fields requiring the understanding of random dynamics. The mathematical formalization of Brownian motion was significantly advanced by Albert Einstein in 1905 [4] and later by Norbert Wiener, who provided a rigorous theoretical framework, henceforth known as the Wiener process [5].

At its core, Brownian motion captures the essence of randomness and its pervasive influence on systems across scales, from microscopic particles to stock market fluctuations.

A simple but descriptive example is the discrete, one-dimensional *random walk* [6]. Consider a particle on the reel axis at $\mathbf{x}_k \in \mathbb{R}$ with $\mathbf{x}_0 = 0$. At each time step $k = 1, 2, \ldots$, it jumps a distance of h either left or right with no preference, that is with probability 1/2 for each side, independent of the previous trajectory. This process can be expressed in terms of a Markovian transition density given by

$$p(\mathbf{x}_k|\mathbf{x}_{k-1}) = \begin{cases} \frac{1}{2}, & \text{if } |\mathbf{x}_k - \mathbf{x}_{k-1}| = h \\ 0, & \text{else.} \end{cases} \tag{16}$$

Or as a recursive formulation, one has that the density at time t_{k+1} is given by

$$p(\mathbf{x}_{k+1}) = \frac{1}{2}p(\mathbf{x}_k + h) + \frac{1}{2}p(\mathbf{x}_k - h). \tag{17}$$

It is well-known that the pdf of this random motion evolves deterministically according to the diffusion equation with a diffusion constant $D = \frac{h^2}{2\tau}$ for $\tau = t_{k+1} - t_k$ [7]:

$$\partial_t p(\mathbf{x}_t) = D\partial_x^2 p(\mathbf{x}_t), \tag{18}$$

Instead of solving the partial differential equation from (18), our goal is to use the superposition principle from quantum computing to represent the density at each time step in terms of the quantum state amplitudes. This idea is called *quantum simulation* and has been studied for quite some time [8], [9]. In the literature, one can find various applications such as graph search [10], optimal stock options pricing [11], or multi dimensional integration [12]. The essential idea is simple and is based on two registers[3]. The first register has n qubits and represents the density function in the state space. The second register has k qubits and is used to 'simulate' the randomness in terms of a superposition of possible outcomes.

a) Single Quantum Transition: Assume, we have initialized a circuit with n qubits in the state $|\mathbf{j}\rangle$ where $0 < j < 2^n - 1$. Since all probability mass is in this state, we want that after one step of the Brownian motion with unit step size $h = 1$ one would obtain the distribution by repeated measurements and obtain $P(j+1) = P(j-1) = 1/2$. Assume, we have given two quantum gates, which can increase and decrease the quantum state, respectively:

$$\text{plus} : |\mathbf{j}\rangle \longrightarrow |\mathbf{j}+1\rangle \tag{19}$$

$$\text{minus} : |\mathbf{j}\rangle \longrightarrow |\mathbf{j}-1\rangle \tag{20}$$

Furthermore, in the second register there is a single qubit, which is in perfect superposition

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle. \tag{21}$$

Now, a controlled application of the plus and minus gates yields a single step of the above described process: A *controlled* gate U, written as CU, appends an extra qubit argument q_c and is defined as

$$CU|q_c, q_t^{(1)}, \ldots, q_t^{(l)}\rangle := \begin{cases} |q_c, U(q_t^{(1)}, \ldots, q_t^{(l)})\rangle & \text{if } q_c = |1\rangle \\ |q_c, q_t^{(1)}, \ldots, q_t^{(l)}\rangle & \text{if } q_c = |0\rangle \end{cases} \tag{22}$$

Here, U is assumed to operate on l target qubits $q_t^{(1)}, \ldots, q_t^{(l)}$. Therefore, one has that

$$C\text{plus}|+, \mathbf{j}\rangle = \frac{1}{\sqrt{2}}|+, \mathbf{j}\rangle + \frac{1}{\sqrt{2}}|+, \mathbf{j}+1\rangle \tag{23}$$

On the other hand, we may apply the minus gate in the case, that the control qubit is in state $|0\rangle$ by inverting it and apply the controlled operation afterwards. Inverting a qubit is achieved by the X Pauli gate, which is represented by the unitarian matrix

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{24}$$

As a consequence, we may combine both operations, which then yields

$$\begin{aligned} &(X \otimes \mathbf{I})C\text{minus}(X \otimes \mathbf{I})C\text{plus}|+, \mathbf{j}\rangle \\ &\qquad = \frac{1}{\sqrt{2}}(|0, \mathbf{j}-1\rangle + |1, \mathbf{j}+1\rangle), \end{aligned} \tag{25}$$

where $\mathbf{I} = I \otimes \cdots \otimes I$ is the identity of proper dimension. The inversion of the control qubit usually is applied afterwards again in order to restore the original state. One can see that we have achieved a superposition of the step up to $\mathbf{j}+1$ and the step down to $\mathbf{j}-1$. This small circuit is shown for $n = 4$ state qubits in Figure 1.

[3] A register is simply a collection of qubits.

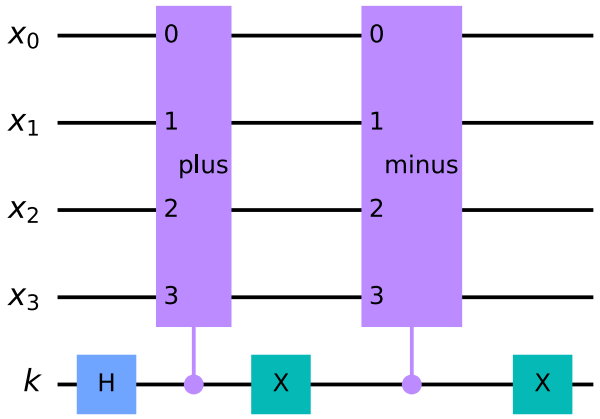

Fig. 1. Quantum circuit for the simulation of a single Brownian motion step with unit step size.

In the figure, the dot-connections of gates indicate the control qubit and H is the *Hadamard* gate

$$H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{with } H|0\rangle = |+\rangle. \tag{26}$$

b) Multi Step Quantum Diffusion: The circuit in Figure 1 is a single step for a random walk of a Brownian motion. The diffusion of a density is obtained by appending multiple steps such that the desired additive variance is obtained. Since an independent source of superposition is required, each step is based on a separate control gate q_c. The fundamental solution of the diffusion differential equation in (18) for $h = 1$ and $\tau = 1$ is given by the normal distribution with stationary mean and the variance increases linear in time. This implies that m steps increases the variance by m.

We may use the circuit in Figure 1 as a gate called "PlusMinus", which uses $n+1$ qubits, such that the control qubit is the last one. Figure 2 shows the resulting exemplary implementation.

V. Integration of a Drift Register

The quantum simulation of the Brownian motion provides the means to represent the stochastic diffusion for the representation of a pdf. The drift function in (2) is deterministic usually applies the laws of classical physics for the considered system. In target tracking for instance, this is the kinematic motion, where the velocity and higher orders have been estimated based on previous information and sensor data.

Let us assume that a third quantum register has been prepared, which represents the pdf of the velocity. A canonical encoding is given as follows. For a velocity register with n_v qubits, a vector $\mathbf{v} \in \mathbb{N}^{N_v}$ with $N_v = 2^{n_v}$ is the velocity state space normalized by time increment τ such that $\tau[\mathbf{v}]_i \in \mathbb{N}$ and the qubits are prepared with $P(\mathbf{j}_v) = [\mathbf{v}]_{\mathbf{j}_v+1}$ is the pdf of the $\mathbf{j}_v + 1$ component of $\mathbf{v}$ for $\mathbf{j}_v \in \{0, \ldots, N_v - 1\}$.

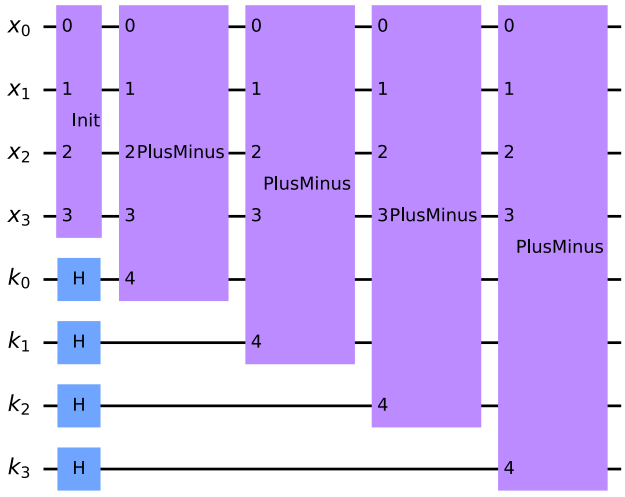

Fig. 2. Quantum circuit for the simulation of a diffusion, which is based on four Brownian motion steps.

As an example consider a register with two qubits such that $N_v = 4$. One may consider positive and negative velocities with equidistant discretization, for instance

$$\mathbf{v} = \begin{pmatrix} -1 & 0 & 1 & 2 \end{pmatrix}^\top. \tag{27}$$

In this example, the amplitude a_{00} for the velocity state $|00\rangle = |\mathbf{0}\rangle$ yields the pdf for the velocity $v = -1$, when the absolute value is squared. Here unit time increment τ is assumed. For other cases, the vector $\mathbf{v}$ has to be adapted, such that it always corresponds to the state increments. Non-integer drift values are obtained by means of the superposition principle of the velocity basis states.

Again, it is sufficient to implement the drift circuit for all basis vectors $|\mathbf{j}_v\rangle$. Since for these the state increments are integer multiples of upshifts ("Plus") or downshifts ("Minus"), they can be implemented by an (multiple) application of the Plus and Minus gates using control qubits in the velocity register. That is, for all basis states $\mathbf{j} = \begin{pmatrix} b_{n_v} & b_{n_v-1} & \ldots & b_1 \end{pmatrix}$ in their binary representation with $b_i \in \{0, 1\}$ we apply the the corresponding controlled operation conditioned on b_i, if $b_i = 1$ and conditioned on $\neg b_i$, if $b_i = 0$. As above, the inversion of a qubit is obtained by the X gate.

For instance, consider again the example from above, where $v = -1$ for $|00\rangle$. Here, both qubits have to be inverted by an $X \otimes X$ operation, then a single controlled Minus gate is applied. Afterwards the inversion is reversed by two X gates again. The full circuit for the velocity vector in (27) is shown in Figure 3. Here the plus^2 gate refers to a double application of the Plus operation.

VI. Numerical Evaluation

In this section, the results of the quantum simulation for Brownian motion, diffusion, and drift is evaluated based on the quantum computation simulator in the IBM

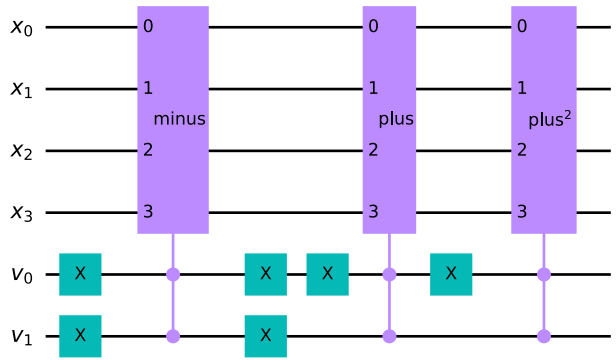

Fig. 3. Quantum circuit for the drift with two qubits and a drift vector given in (27).

Qiskit framework [13]. After each step, the distribution is obtained by repeated measurements. For the next step, the qubits are prepared in the corresponding states again.

The numerical results of an implementation of the Brownian motion described above with $n = 4$ qubits is shown in Figure 4. Here, the process is initialized in the basis state $|7\rangle$, which approximately is in the center of the state space $\{0, 1, \ldots, 15\}$. One can see the "holes" between to successive states, which are a direct consequence of the definition for the Brownian motion. Another evaluation of the resulting probabilities for the Brownian motion is shown in Figure 5, where the initialization is given by a normal distribution $\mathcal{N}\big(x; 7, 1\big)$. Here, the holes are not present anymore.

The mean and standard deviations for the drift and diffusion quantum simulation are given in Figure 6 for different velocities. One can see that the variance increases for each step by one, that is, the standard deviation is given by $\sqrt{k+1}$, where k is the time step, which is initialized with $\sigma = 1$ for $k = 0$. A spillover effect can be observed in Figure 6 (d) for $v = 2$, where some of the probability mass has swapped beyond the upper limit of $\mathbf{j} = 15$ and is rolled to $\mathbf{j} = 0$. As a consequence, the resulting mean is slightly lower and the variance slightly higher.

VII. Conclusion

In this paper, we used the concept of quantum simulation to solve the application of a continuous noise motion model to a probability density function (pdf). The pdf is prepared as the squared amplitudes for the basis states spanned by the number of used qubits. The number of required qubits grows linear in the number of time steps to simulate in a prediction phase. One step performs a single Brownian motion, which is used to generate the diffusion of the Wiener increments. A drift function can be implemented based on a separate register, which holds the pdf on the velocity information for an adequate discretization. The approach has been visualized in terms of quantum circuits and evaluated based on a quantum simulator.

Appendix

Plus Gate

The Plus gate increases the basis state by one such that

$$\text{Plus}\,|\mathbf{j}\rangle = |\mathbf{j}+1\rangle. \tag{28}$$

In order to setup the quantum circuit, one may look at the mapping in terms of a binary table and observe the occurring flips for each qubit. Consider for instance the case of $n = 3$ qubits given in Table I.

0	0	0	0	0	1
0	0	1	0	1	0
0	1	0	0	1	1
0	1	1	1	0	0
1	0	0	1	0	1
1	0	1	1	1	0
1	1	0	1	1	1
1	1	1	0	0	0

TABLE I
Binary mapping table of the Plus gate for three qubits.

The operation on the basis states is given in Table I. As one can see, the plus operation is fully described by the following two rules:

- The first qubit q_1 is always inverted.
- A qubit q_l for $l > 1$ is inverted, if and only if all qubits $q_1, \ldots, q_{l-1}$ are one.

Obviously, this can easily be implemented into a circuit using the "not gate" X and controlled versions of it. An example for $n = 4$ qubits is given in Figure 7 (a).

Minus Gate

All gates and therefore circuits are unitary operations. Since the Minus gate is the inverse of the Plus gate, we may simply derive its inverse from the derivations above. One can see that the X gate and its controlled versions are self inverting, that is, they form the identity when applied twice. As a consequence, we can set up a Plus gate in inverse order and obtain a Minus gate such that Plus Minus $= \mathbf{I}$. The example for $n = 4$ is shown in Figure 7 (b).

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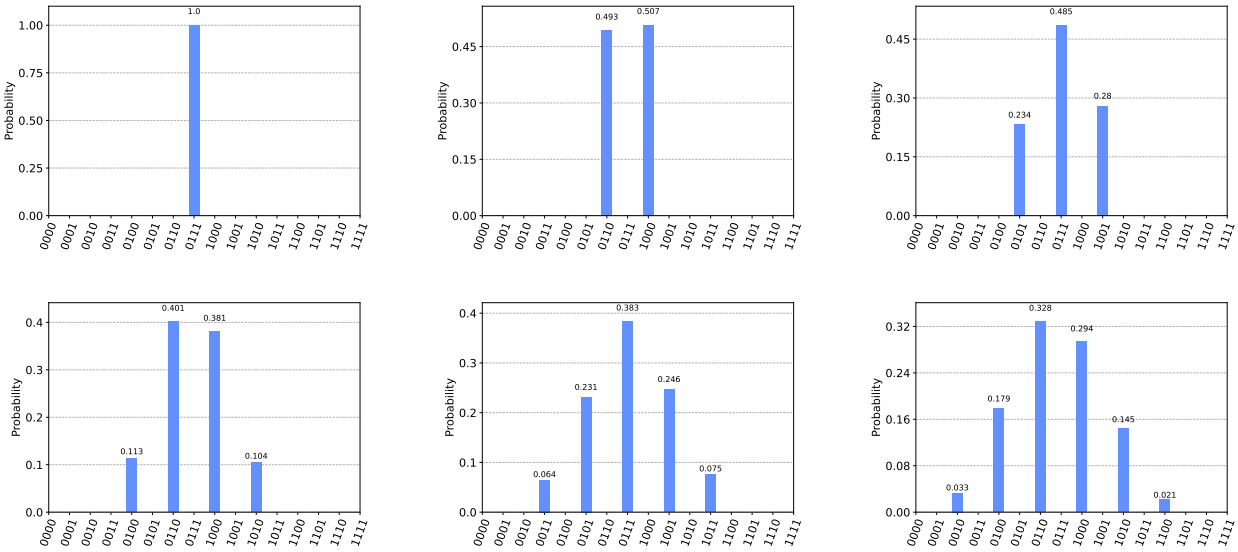

Fig. 4. Probability distribution for $n = 4$ qubits and $N = 16$ states of a Brownian motion. The process is initialized with $|7\rangle$.

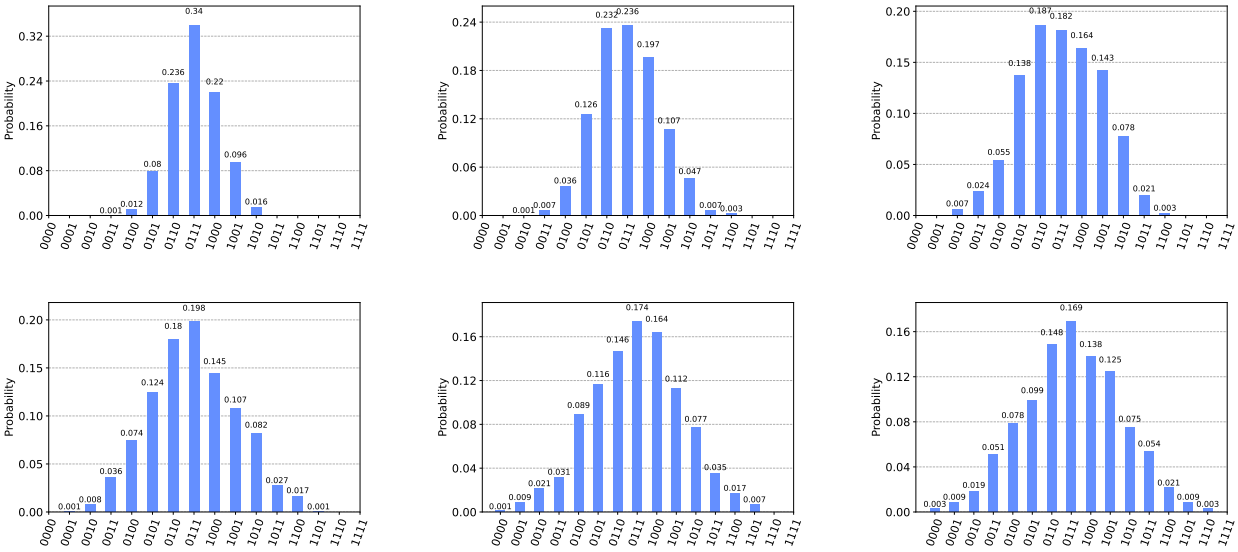

Fig. 5. Probability distribution for $n = 4$ qubits and $N = 16$ states of a Brownian motion. The process is initialized with a normal distribution with mean 7 and variance 1.

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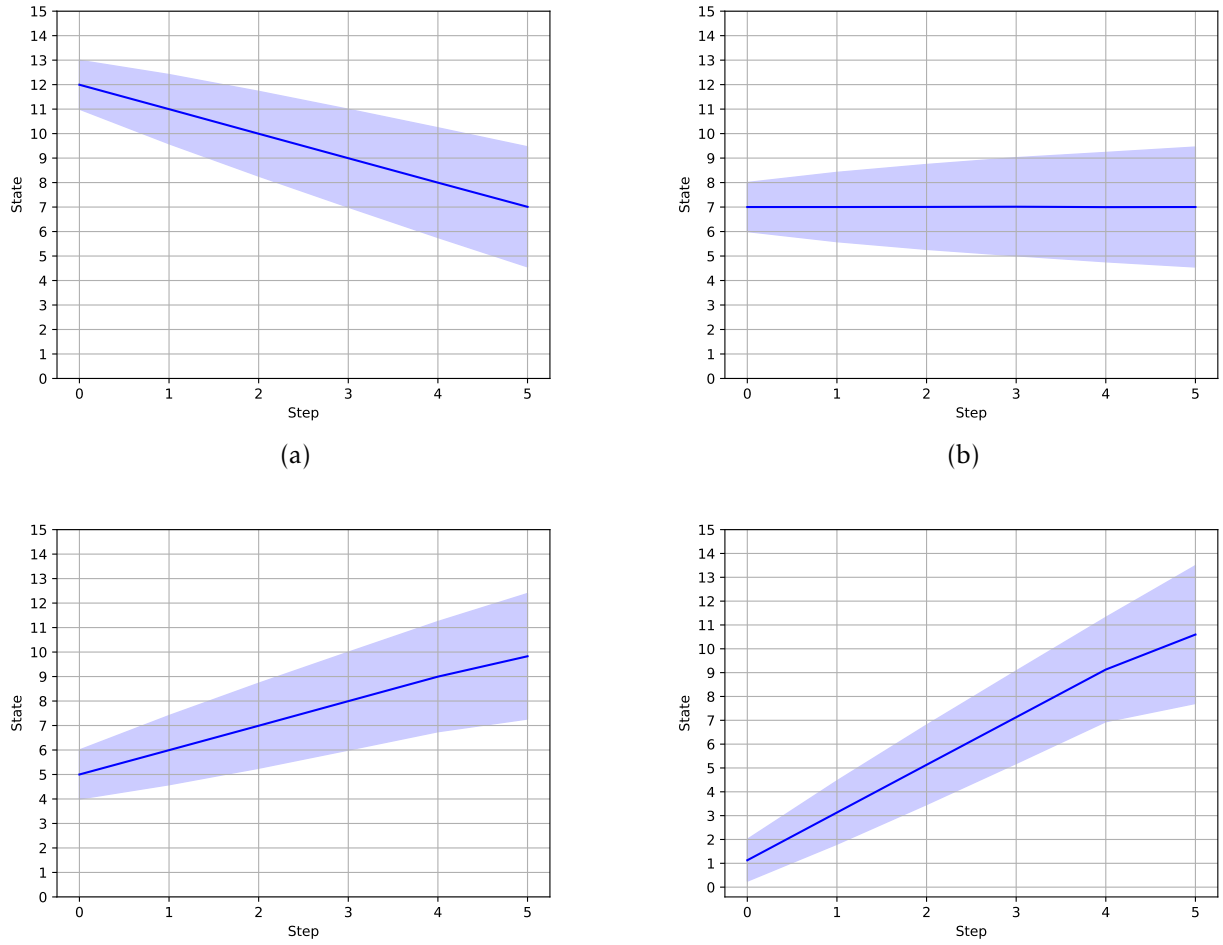

Fig. 6. Mean and standard deviation of the pdf predicted for multiple time steps for velocities -1 (a), 0 (b), 1 (c), and 2 (d).

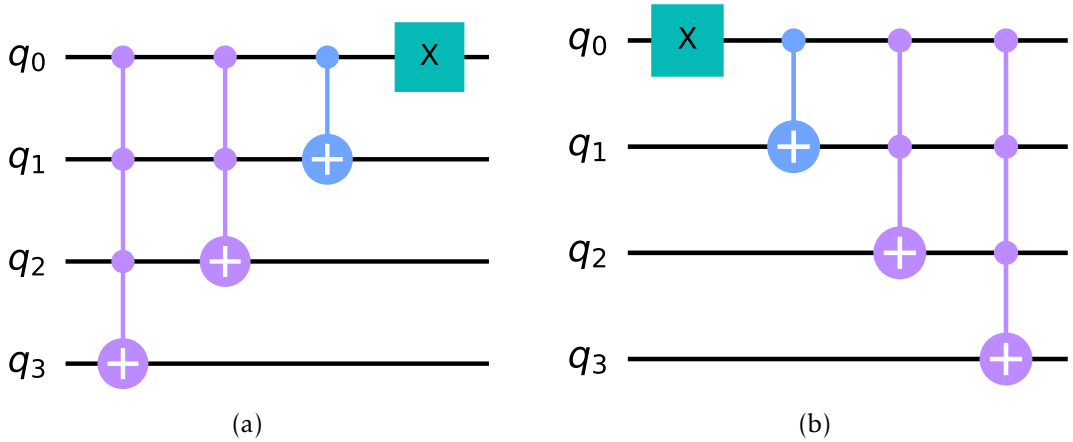

Fig. 7. Quantum circuit for the plus $|\mathbf{j}\rangle \rightarrow |\mathbf{j}+1\rangle$ (a) and minus $|\mathbf{j}\rangle \rightarrow |\mathbf{j}-1\rangle$ (b) operation.